8.311 Recitation Notes

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I. INTRODUCTION

Consider the matrix multiplication $\boldsymbol{A} \cdot \boldsymbol{B}$; what is this in coordinates? We have that:

$$(\boldsymbol{A} \cdot \boldsymbol{B})_{ij} = \sum_k A_{ik} B^k{}_j = A_{ik} B^k{}_j. \tag{1}$$

The dropping of the summation symbol is called the *Einstein summation convention.* In it, repeated indices are denoted "dummy" indices and are summed over; all other indices are denoted "free" indices, and give the component indices of the resulting symbol.

Some examples of common operations are:

$$(\boldsymbol{a} \otimes \boldsymbol{b})_{ij} = a_i b_j, \tag{2}$$

$$\mathrm{tr}\,(\boldsymbol{A}) = A^i{}_i, \tag{3}$$

$$\mathrm{tr}\,(\boldsymbol{A} \cdot \boldsymbol{B}) = A^i{}_j B^j{}_i, \tag{4}$$

$$\mathrm{tr}\,(\boldsymbol{A} \cdot \boldsymbol{B} \cdot \boldsymbol{C}) = A^i{}_j B^j{}_k B^k{}_i. \tag{5}$$

Eq. (5) makes the cyclic property of the trace manifest.

Now, let us introduce basic symbols that are the building blocks of various index manipulations. An important symbol is the *Kronecker delta*, with components:

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}; \tag{6}$$

as a matrix, this is just the identity matrix.

Another important symbol is the *Levi-Civita symbol*, with components:

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } ijk = 123, 231, 312 \\ -1, & \text{if } ijk = 132, 213, 321 \\ 0, & \text{otherwise} \end{cases}. \tag{7}$$

It is defined by its complete antisymmetry, which means it is antisymmetric upon the exchange of any two of its indices; that is,

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}, \tag{8}$$

which also gives its cyclic property:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}. \tag{9}$$

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The Levi-Civita symbol is important as it allows a concise description of cross products:

$$(\boldsymbol{A} \times \boldsymbol{B})_i = \epsilon_{ijk} A^j B^k. \tag{10}$$

An important identity involving both the Kronecker delta and the Levi-Civita symbol is:

$$\epsilon_{ijk} \epsilon^i{}_{nm} = \delta_{jn} \delta_{km} - \delta_{jm} \delta_{kn}. \tag{11}$$

An easy way to remember the signs of this identity are that "like indices" yield a positive sign and "cross indices" yield a negative sign.

II. CHANGING BASES

With the developed machinery, how can we transform vectors between different bases? Consider a change of basis matrix $\boldsymbol{R}$, defined such that when changing from a basis $\left\{\hat{\boldsymbol{e}}_{(i)}\right\}_i$ to another basis $\left\{\hat{\boldsymbol{e}}'_{(i)}\right\}_i$,

$$\hat{\boldsymbol{e}}'_{(i)} = R_i{}^j \hat{\boldsymbol{e}}_{(j)}; \tag{12}$$

componentwise,

$$R_i{}^j = \hat{\boldsymbol{e}}'_{(i)} \cdot \hat{\boldsymbol{e}}^{(j)}. \tag{13}$$

This gives the transformation of a vector

$$\boldsymbol{x} = x^i \hat{\boldsymbol{e}}_{(i)} \tag{14}$$

to a basis $\left\{\hat{\boldsymbol{e}}'_{(i)}\right\}_i$ to be

$$\boldsymbol{x}' = (x')^i \, \hat{\boldsymbol{e}}'_{(i)} = (x')^i \, R_i{}^j \hat{\boldsymbol{e}}_{(j)}, \tag{15}$$

which gives componentwise:

$$x^j = (x')^i \, R_i{}^j; \tag{16}$$

equivalently,

$$(x')^i = x^j \left(R^{-1}\right)_j{}^i. \tag{17}$$

As $\left(R^{-1}\right)_{ij} = R_{ji}$ (which you will show on your problem set), this can be expressed neatly as the matrix multiplication (where $\boldsymbol{x}$ and $\boldsymbol{x}'$ are column vectors):

$$\boldsymbol{x}' = \boldsymbol{R} \cdot \boldsymbol{x}. \tag{18}$$

This is called a *contravariant* transformation law, as the transformation of vectors is done by the inverse change of basis matrix R^{-1}. In Einstein notation, vectors that transform contravariantly ("contravariant vectors") have indices "upstairs"—in matrix notation, these are column vectors. Examples of contravariant vectors include coordinates in position and velocity space and kets. Conversely, row vectors ("covariant vectors") follow a *covariant* transformation law:

$$\boldsymbol{x'} = \boldsymbol{x} \cdot \boldsymbol{R^{-1}}. \tag{19}$$

Examples of covariant vectors include the spatial gradient vector, the inverse position vector, and bras. For completeness, scalars are considered "invariant", as they do not transform under changes of basis. The transformation of an object as given by Eq. (18) under a change of basis therefore *defines* contravariant vectors, and similarly the transformation of an object as given by Eq. (19) *defines* covariant vectors. An easy way to see that they must transform oppositely is to consider how the inner product $\boldsymbol{x} \cdot \boldsymbol{y}$ transforms under a change of basis; we have that:

$$\boldsymbol{x'} \cdot \boldsymbol{y'} = \boldsymbol{x} \cdot \boldsymbol{R^{-1}} \cdot \boldsymbol{R} \cdot \boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y}. \tag{20}$$

That is, inner products are preserved under changes of basis; this must be true, as changes of basis (in real space) in n-dimensions form the group $\mathrm{O}\left(n\right)$, which preserves inner products.

As an example, let us consider the change of basis from the standard basis

$$\left\{\boldsymbol{\hat{e}}_{(i)}\right\} = \left\{\boldsymbol{\hat{x}}, \boldsymbol{\hat{y}}, \boldsymbol{\hat{z}}\right\} \tag{21}$$

to the basis

$$\left\{\boldsymbol{\hat{e}'}_{(i)}\right\} = \left\{\cos\left(\theta\right)\boldsymbol{\hat{x}} + \sin\left(\theta\right)\boldsymbol{\hat{y}}, -\sin\left(\theta\right)\boldsymbol{\hat{x}} + \cos\left(\theta\right)\boldsymbol{\hat{y}}, \boldsymbol{\hat{z}}\right\}. \tag{22}$$

This is a counterclockwise rotation of the standard xy-basis about the z-axis; this is easy to see by checking what happens when $\theta = \frac{\pi}{2}$. From Eq. (13), we have that

$$R_i^{\ j}\left(\theta\right) = \begin{pmatrix} \cos\left(\theta\right) & \sin\left(\theta\right) & 0 \\ -\sin\left(\theta\right) & \cos\left(\theta\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}_i^{\ j}. \tag{23}$$

Eq. (18) then gives the general transformation law for column vectors (and generally, contravariant vectors) and Eq. (19) the general transformation law for row vectors (and generally, covariant vectors). For instance, $\frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\right)$ becomes the vector:

$$\boldsymbol{R}\left(\frac{\pi}{2}\right)\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\\ 0\end{pmatrix} \tag{24}$$

after a rotation of the basis vectors by $\frac{\pi}{2}$ counterclockwise, which is easy to check (note this is *not* a rotation of the vector itself by $\frac{\pi}{2}$, which would give opposite signs for the vector's components).

III. GENERAL TENSORS

These notions can be extended to general tensors. The general transformation law for a general tensor is:

$$T_{i'_1\ldots i'_n}{}^{j'_1\ldots j'_m}=R_{i'_n}{}^{i_n}\ldots R_{i'_1}{}^{i_1}T_{i_1\ldots i_n}{}^{j_1\ldots j_m}\left(R^{-1}\right)_{j_1}{}^{j'_1}\ldots\left(R^{-1}\right)_{j_m}{}^{j'_m}; \tag{25}$$

this transformation law *defines* tensors.