8.311 Recitation Notes

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I. INTRODUCTION

In class, we've discussed (or soon will discuss) the *Maxwell stress tensor*:

$$\sigma_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)\delta_{ij}; \tag{1}$$

the *Poynting vector*:

$$S_i = \frac{1}{\mu_0}\epsilon_{ijk} E^j B^k; \tag{2}$$

and the energy density:

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right). \tag{3}$$

These were given derivations in class, but is there a way they arise more fundamentally?

II. NOTATION

I will use the notation where Latin letters denote spatial indices, and Greek letters denote spatial indices in addition to a time index; for instance,

$$\partial_i a^i = \frac{\partial a^1}{\partial x^1} + \frac{\partial a^2}{\partial x^2} + \frac{\partial a^3}{\partial x^3}, \tag{4}$$

and

$$\partial_\mu a^\mu = \frac{\partial a^0}{\partial x^0} + \frac{\partial a^1}{\partial x^1} + \frac{\partial a^2}{\partial x^2} + \frac{\partial a^3}{\partial x^3}. \tag{5}$$

where x^0 is the time(like) coordinate (i.e. $t = \frac{1}{c}x^0$).

Furthermore, I will use the *Minkowski metric*

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}^{\mu\nu} \tag{6}$$

to implicitly transform between contravariant and covariant tensors (i.e. raise and lower indices). That is,

$$a_\mu = \eta_{\mu\nu} a^\nu \tag{7}$$

For instance,

$$\partial_\mu \partial^\mu = \eta^{\mu\nu}\partial_\mu \partial_\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = -\Box. \tag{8}$$

I'll try to hide this so you can follow without it, but you'll need it if you're following carefully.

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III. LAGRANGIAN FORMALISM OF ELECTROMAGNETISM

A. The Electromagnetic Lagrangian

Let us combine the vector potential $\boldsymbol{A}$ and the electric potential ϕ into a single four-vector potential:

$$A^\mu = \begin{pmatrix} \frac{\phi}{\mathrm{c}} \\ \boldsymbol{A} \end{pmatrix}^\mu ; \tag{9}$$

the factor of c is just an artifact of SI units. It turns out that the only Lagrangian density describing massless A^μ that is consistent with quantum mechanics and special relativity is

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu, \tag{10}$$

where F is the *electromagnetic tensor*

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{11}$$

Using Maxwell's equations, one can show that in terms of electric and magnetic fields:

$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{E_1}{\mathrm{c}} & \frac{E_2}{\mathrm{c}} & \frac{E_3}{\mathrm{c}} \\ -\frac{E_1}{\mathrm{c}} & 0 & -B_3 & B_2 \\ -\frac{E_2}{\mathrm{c}} & B_3 & 0 & -B_1 \\ -\frac{E_3}{\mathrm{c}} & -B_2 & B_1 & 0 \end{pmatrix}_{\mu\nu} . \tag{12}$$

Equivalently,

$$F^\mu{}_\nu = \begin{pmatrix} 0 & \frac{E_1}{\mathrm{c}} & \frac{E_2}{\mathrm{c}} & \frac{E_3}{\mathrm{c}} \\ \frac{E_1}{\mathrm{c}} & 0 & B_3 & -B_2 \\ \frac{E_2}{\mathrm{c}} & -B_3 & 0 & B_1 \\ \frac{E_3}{\mathrm{c}} & B_2 & -B_1 & 0 \end{pmatrix}^\mu{}_\nu \tag{13}$$

and

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_1}{\mathrm{c}} & -\frac{E_2}{\mathrm{c}} & -\frac{E_3}{\mathrm{c}} \\ \frac{E_1}{\mathrm{c}} & 0 & -B_3 & B_2 \\ \frac{E_2}{\mathrm{c}} & B_3 & 0 & -B_1 \\ \frac{E_3}{\mathrm{c}} & -B_2 & B_1 & 0 \end{pmatrix}^{\mu\nu} . \tag{14}$$

J^μ just combines the current density $\boldsymbol{J}$ and the charge density ρ into a single four-vector current:

$$J^\mu = \begin{pmatrix} \mathrm{c}\rho \\ \boldsymbol{J} \end{pmatrix}^\mu , \tag{15}$$

and charge conservation is just a statement that

$$\partial_\mu J^\mu = 0. \tag{16}$$

As Lagrangians completely define a field theory, any symmetries of the Lagrangian are symmetries of the theory. Thus, we can naturally see that electromagnetism is gauge invariant; that is, $\mathcal{L}$ is symmetric under $g_\alpha : A_\mu \mapsto A_\mu + \partial_\mu \alpha$. For instance, taking α such that

$$-\partial_\mu \partial^\mu \alpha = \partial^\mu A_\mu \tag{17}$$

sets the Lorenz gauge condition

$$\partial_\mu A^\mu = 0. \tag{18}$$

B. Maxwell's Equations

The Euler–Lagrange equations applied to Eq. (10) gives:

$$\begin{aligned} \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} - \frac{\delta \mathcal{L}}{\delta A_\nu} &= 0 \\ \implies -\frac{1}{4\mu_0} \partial_\mu \left(2\partial^\mu A^\nu - 2\partial^\nu A^\mu - 2\partial^\nu A^\mu + 2\partial^\mu A^\nu\right) + J^\nu &= 0 \\ \implies \partial_\mu F^{\mu\nu} &= \mu_0 J^\nu. \end{aligned} \tag{19}$$

We see that this gives rise to the Maxwell equations

$$\begin{aligned} \partial_\mu F^{\mu 0} &= \mu_0 J^0 \\ \implies \frac{\boldsymbol{\nabla} \cdot \mathbf{E}}{\mathrm{c}} &= \mathrm{c}\mu_0 \rho \\ \implies \boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \end{aligned} \tag{20}$$

and

$$\begin{aligned} \partial_\mu F^{\mu i} &= \mu_0 J^i \\ \implies \boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right). \end{aligned} \tag{21}$$

Meanwhile, Eq. (11) immediately gives rise to the *Bianchi identity*:

$$\partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu} = 0. \tag{22}$$

We see that this gives rise to the Maxwell equations

$$\begin{aligned} \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} &= 0 \\ \implies \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \end{aligned} \tag{23}$$

and (for $i \neq j$)

$$\begin{aligned} &\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0 \\ &\implies \boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \end{aligned} \tag{24}$$

We can also easily derive the relation between the potentials and currents we derive in class. In the Lorenz gauge, Eq. (19) is equivalent to:

$$\partial_\mu \partial^\mu A^\nu = \mu_0 J^\nu \iff \Box A^\mu = -\mu_0 J^\mu, \tag{25}$$

which we inverted in class to find $\boldsymbol{A}$ and ϕ in terms of $\boldsymbol{J}$ and ρ.

IV. THE ELECTROMAGNETIC STRESS-ENERGY TENSOR

By Noether's theorem, spacetime invariance gives rise to the stress-energy tensor; Noether's theorem gives (for $J^\mu = 0$) the canonical stress-energy tensor:

$$\begin{aligned} T^{\mu\nu}_{\text{canonical}} &= \partial^\mu \left(A^\alpha\right) \frac{\delta \mathcal{L}}{\delta \partial_\nu A^\alpha} - \eta^{\mu\nu} \mathcal{L} \\ &= -\frac{1}{\mu_0} F^{\nu\alpha} \partial^\mu A_\alpha + \frac{1}{4\mu_0} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}. \end{aligned} \tag{26}$$

Symmetrizing $T^{\mu\nu}_{\text{canonical}}$ to take account of Lorentz transformations (or alternatively finding the stress-energy tensor by varying the metric), this is equivalent to the symmetric stress-energy tensor

$$T^{\mu\nu} = \frac{1}{\mu_0} \left(F^\mu{}_\alpha F^{\alpha\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right). \tag{27}$$

Using Eq. (11) we therefore have that:

$$T^{\mu\nu} = \begin{pmatrix} u & \frac{S_1}{c} & \frac{S_2}{c} & \frac{S_3}{c} \\ \frac{S_1}{c} & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ \frac{S_2}{c} & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\ \frac{S_3}{c} & -\sigma_{31} & -\sigma_{32} & -\sigma_{33} \end{pmatrix}^{\mu\nu}. \tag{28}$$

This immediately gives the intuition behind u, $\boldsymbol{S}$, and $\boldsymbol{\sigma}$ (or, alternatively, gives intuition for the stress-energy tensor). $T^{00} = u$ is the energy density, $T^{0i} = \frac{S^i}{c}$ is the momentum density, and $-T^{ij} = \sigma_{ij}$ is the stress tensor.