8.311 Recitation Notes

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(Dated: February 21, 2019)

I. INTRODUCTION

In lecture (and on your problem set), you explicitly took the divergence and curl to show that

$$\boldsymbol{F}(\boldsymbol{r},t) = -\frac{1}{4\pi}\boldsymbol{\nabla}\int \mathrm{d}^3r' \, \frac{\boldsymbol{\nabla}\cdot\boldsymbol{F}(\boldsymbol{r}',t)}{|\boldsymbol{r}-\boldsymbol{r}'|} + \frac{1}{4\pi}\boldsymbol{\nabla}\times\int \mathrm{d}^3r' \, \frac{\boldsymbol{\nabla}\times\boldsymbol{F}(\boldsymbol{r}',t)}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{1}$$

assuming F decays quickly enough as $r \to \infty$. Here, we will go over some basic properties of Fourier transforms and use them to give an alternative derivation of Helmholtz's theorem.

II. THE FOURIER TRANSFORM

Let us first prove a useful identity for the Dirac delta function. Consider the function

$$f_a(x) = \int \mathrm{d}k \, \mathrm{e}^{\mathrm{i}k(x-a)}. \tag{2}$$

For $x \neq a$, the integral averages over all oscillations to zero such that

$$f_a(x) = 0. \tag{3}$$

However, for $x \to a$, $f_a(x)$ becomes an integral over all k of 1 such that

$$\lim_{x\to a} f_a(x) = \infty. \tag{4}$$

Furthermore, $f_a(x)$ is normalized such that

$$\begin{aligned}
\int \mathrm{d}x \, f_a(x) &= \int \mathrm{d}x \int \mathrm{d}k \, \mathrm{e}^{\mathrm{i}k(x-a)} \\
&= -\mathrm{i}\int \mathrm{d}k \, \frac{\mathrm{e}^{\mathrm{i}k(x-a)}}{k}\Bigg|_{x=-\infty}^{\infty} \\
&= -\mathrm{i}\int \mathrm{d}k \left(\frac{\cos(k(x-a)) + \mathrm{i}\sin(k(x-a))}{k}\right)\Bigg|_{x=-\infty}^{\infty} \\
&= \int \mathrm{d}k \, \frac{\sin(k(x-a))}{k}\Bigg|_{x=-\infty}^{\infty} \\
&= \lim_{x\to\infty}(\pi\,\mathrm{sgn}(x-a)) - \lim_{x\to-\infty}(\pi\,\mathrm{sgn}(x-a)) \\
&= 2\pi.
\end{aligned} \tag{5}$$

Therefore,

$$2\pi\delta(x-a) = f_a(x) = \int \mathrm{d}k \, \mathrm{e}^{\mathrm{i}k(x-a)}. \tag{6}$$

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We therefore notice that:

$$\begin{aligned}\frac{1}{2\pi}\int \mathrm{d}k\left(\int \mathrm{d}y\, f\left(y\right)\mathrm{e}^{-\mathrm{i}ky}\right)\mathrm{e}^{\mathrm{i}kx} &= \frac{1}{2\pi}\int \mathrm{d}y\int \mathrm{d}k\, f\left(y\right)\mathrm{e}^{\mathrm{i}k\left(x-y\right)} \\ &= \int \mathrm{d}y\, f\left(y\right)\delta\left(x-y\right) \\ &= f\left(x\right).\end{aligned} \tag{7}$$

This motivates our definition of the *Fourier transform* of a function $f\left(x\right)$, which is given by:

$$\hat{f}\left(k\right) = \int \mathrm{d}x\, f\left(x\right)\mathrm{e}^{-\mathrm{i}kx}. \tag{8}$$

By Eq. (7), we therefore have that the *inverse Fourier transform* of a function $\hat{f}\left(k\right)$ is given by:

$$f\left(x\right) = \frac{1}{2\pi}\int \mathrm{d}k\, \hat{f}\left(k\right)\mathrm{e}^{\mathrm{i}kx}. \tag{9}$$

Note that many different conventions for the Fourier transform and inverse Fourier transform exist, with different factors of 2π (all meant to cancel the 2π normalization appearing in Eq. (5)). For d dimensions, one can take the Fourier transform of each component of $\boldsymbol{r}$ and $\boldsymbol{k}$ to give the Fourier transform:

$$\hat{f}\left(\boldsymbol{k}\right) = \int \mathrm{d}^d r\, f\left(\boldsymbol{r}\right)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \tag{10}$$

and the inverse Fourier transform:

$$f\left(\boldsymbol{r}\right) = \frac{1}{\left(2\pi\right)^d}\int \mathrm{d}^d k\, \hat{f}\left(\boldsymbol{k}\right)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{11}$$

III. THE CONVOLUTION THEOREM

Given a function $\hat{h}\left(\boldsymbol{k}\right) = \hat{f}\left(\boldsymbol{k}\right)\hat{g}\left(\boldsymbol{k}\right)$, one can ask: what is the inverse Fourier transform h of $\hat{h}$?

To answer this question, let us consider the *convolution* operation:

$$\left(f * g\right)\left(\boldsymbol{r}\right) = \int \mathrm{d}^d r'\, g\left(\boldsymbol{r}'\right) f\left(\boldsymbol{r}-\boldsymbol{r}'\right). \tag{12}$$

In terms of the Fourier transforms $\hat{f}$ and $\hat{g}$, we have that:

$$\begin{aligned}
(f * g)(\boldsymbol{r}) &= \int \mathrm{d}^d r'\, g(\boldsymbol{r}')\, f(\boldsymbol{r}-\boldsymbol{r}') \\
&= \frac{1}{(2\pi)^d} \int \mathrm{d}^d r'\, g(\boldsymbol{r}') \int \mathrm{d}^d k\, \hat{f}(\boldsymbol{k})\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')} \\
&= \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \left(\int \mathrm{d}^d r'\, g(\boldsymbol{r}')\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'} \right) \hat{f}(\boldsymbol{k})\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \\
&= \frac{1}{(2\pi)^d} \int \mathrm{d}^d k\, \hat{f}(\boldsymbol{k})\, \hat{g}(\boldsymbol{k})\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}.
\end{aligned} \tag{13}$$

Therefore, if $\hat{h}(\boldsymbol{k}) = \hat{f}(\boldsymbol{k})\, \hat{g}(\boldsymbol{k})$, the inverse Fourier transform h of $\hat{h}$ is:

$$h(\boldsymbol{r}) = (f * g)(\boldsymbol{r}). \tag{14}$$

Note that this construction is only valid if f and g drop off rapidly enough as $r \to \infty$ such that the convolution integral is well-defined.

IV. A SHORTCUT TO THE HELMHOLTZ'S THEOREM

Let us now give an alternative proof of Helmholtz's theorem using the Fourier transform. Let us consider the spatial Fourier transform $\boldsymbol{G}$ of $\boldsymbol{F}$; that is,

$$\boldsymbol{F}(\boldsymbol{r}, t) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k\, \boldsymbol{G}(\boldsymbol{k}, t)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{15}$$

Now, let us consider $\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b})$; expanding the cross products, we find that:

$$\begin{aligned}
(\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b}))^i &= \epsilon^{ijk} a_j \epsilon_{klm} a^l b^m \\
&= \epsilon^{kij} \epsilon_{klm} a_j a^l b^m \\
&= \left(\delta^i{}_l \delta^j{}_m - \delta^i{}_m \delta^j{}_l \right) a_j a^l b^m \\
&= a^i a_j b^j - b^i a_j a^j \\
&= (\boldsymbol{a} \cdot \boldsymbol{b})\, a^i - (\boldsymbol{a} \cdot \boldsymbol{a})\, b^i.
\end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned}
\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{G}) &= (\boldsymbol{k} \cdot \boldsymbol{G})\, \boldsymbol{k} - k^2 \boldsymbol{G} \\
\implies \boldsymbol{G} &= \frac{1}{k^2} \left((\boldsymbol{k} \cdot \boldsymbol{G})\, \boldsymbol{k} - \boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{G}) \right).
\end{aligned} \tag{17}$$

Therefore,

$$\begin{aligned}
\boldsymbol{F}(\boldsymbol{r},t) &= \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, \boldsymbol{G}(\boldsymbol{k},t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \\
&= \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, \frac{1}{k^2}\left(\left(\boldsymbol{k}\cdot\boldsymbol{G}(\boldsymbol{k},t)\right)\boldsymbol{k} - \boldsymbol{k}\times\left(\boldsymbol{k}\times\boldsymbol{G}(\boldsymbol{k},t)\right)\right)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \\
&= -\boldsymbol{\nabla}\int \mathrm{d}^3k\, \frac{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{G}(\boldsymbol{k},t)}{(2\pi)^3 k^2}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} + \boldsymbol{\nabla}\times\left(\int \mathrm{d}^3k\, \frac{\mathrm{i}\boldsymbol{k}\times\boldsymbol{G}(\boldsymbol{k},t)}{(2\pi)^3 k^2}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right).
\end{aligned} \tag{18}$$

This is, in a sense, already Helmholtz's theorem; it gives a dcomposition of $\mathbf{F}$ into a gradient part and a curl part. To get this decomposition in the familiar form given in lecture, we can apply the convolution theorem. We therefore have that:

$$\begin{aligned}
\boldsymbol{F}(\boldsymbol{r},t) = -\boldsymbol{\nabla}&\left(\left(\int \mathrm{d}^3k\, \frac{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{G}(\boldsymbol{k},t)}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right) * \left(\int \mathrm{d}^3k\, \frac{1}{(2\pi)^3 k^2}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right)\right) \\
+ \boldsymbol{\nabla}\times&\left(\left(\int \mathrm{d}^3k\, \frac{\mathrm{i}\boldsymbol{k}\times\boldsymbol{G}(\boldsymbol{k},t)}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right) * \left(\int \mathrm{d}^3k\, \frac{1}{(2\pi)^3 k^2}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right)\right).
\end{aligned} \tag{19}$$

As the inverse Fourier transform of $\hat{f}(\boldsymbol{k}) = \frac{1}{k^2}$ is

$$\begin{aligned}
f(\boldsymbol{r}) &= \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, \frac{1}{k^2}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \\
&= \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}k \int_{-1}^{1} \mathrm{d}\left(\cos(\theta)\right)\mathrm{e}^{\mathrm{i}kr\cos(\theta)} \\
&= \frac{1}{2\pi^2 r}\int_0^\infty \mathrm{d}k\, \frac{\sin(kr)}{k} \\
&= \frac{1}{4\pi r},
\end{aligned} \tag{20}$$

we therefore have that:

$$\begin{aligned}
\boldsymbol{F}(\boldsymbol{r},t) &= -\boldsymbol{\nabla}\left(\left(\int \mathrm{d}^3k\, \frac{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{G}(\boldsymbol{k},t)}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right) * \frac{1}{4\pi r}\right) \\
&\quad + \boldsymbol{\nabla}\times\left(\left(\int \mathrm{d}^3k\, \frac{\mathrm{i}\boldsymbol{k}\times\boldsymbol{G}(\boldsymbol{k},t)}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right) * \frac{1}{4\pi r}\right) \\
&= -\frac{1}{4\pi}\boldsymbol{\nabla}\int \mathrm{d}^3r'\left(\int \mathrm{d}^3k\, \frac{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{G}(\boldsymbol{k},t)}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'}\right)\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \\
&\quad + \frac{1}{4\pi}\boldsymbol{\nabla}\times\int \mathrm{d}^3r'\left(\int \mathrm{d}^3k\, \frac{\mathrm{i}\boldsymbol{k}\times\boldsymbol{G}(\boldsymbol{k},t)}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'}\right)\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}.
\end{aligned} \tag{21}$$

Finally, as

$$\boldsymbol{\nabla}\cdot\boldsymbol{F}(\boldsymbol{r},t) = \frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^3k\, \boldsymbol{k}\cdot\boldsymbol{G}(\boldsymbol{k},t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \tag{22}$$

and

$$\boldsymbol{\nabla} \times \boldsymbol{F}(\boldsymbol{r}, t) = \frac{\mathrm{i}}{(2\pi)^3} \int \mathrm{d}^3k\, \boldsymbol{k} \times \boldsymbol{G}(\boldsymbol{k}, t)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{23}$$

we have exactly Helmholtz's theorem as given in Eq. (1), assuming F decays quickly enough as $r \to \infty$ such that the convolution theorem holds.