8.311 Recitation Notes

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I. INTRODUCTION

This recitation will be a review of material that could appear on the first exam. The material the exam will cover is the same as the first three problem sets, and the problems will be in the style of those in the problem sets.

II. MAXWELL'S EQUATIONS

You should know these well by now:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{3}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right), \tag{4}$$

along with the Lorentz force law:

$$\boldsymbol{f} = \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}, \tag{5}$$

where:

$$\boldsymbol{f} = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}V}. \tag{6}$$

Maxwell's equations and the Helmholtz decomposition (see Sec. IV) give us the *scalar potential* ϕ and the *vector potential* $\boldsymbol{A}$, which satisfy:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{7}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{8}$$

$\boldsymbol{A}$ is defined up to a gradient (i.e. one can take its divergence to be whatever one finds convenient), as this term can be absorbed into the $-\boldsymbol{\nabla}\phi$ term for $\boldsymbol{E}$ and disappears in the curl for $\boldsymbol{B}$. Important gauges are the *Lorenz gauge*, where:

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = -\epsilon_0 \mu_0 \frac{\partial \phi}{\partial t}, \tag{9}$$

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and the *Coulomb gauge*, where:

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0. \tag{10}$$

You do not need to know the Coulomb gauge as it was not covered in the lectures, but it is a good thing to know in general.

Finally, a useful identity to know is:

$$\frac{1}{\epsilon_0 \mu_0} = \mathrm{c}^2. \tag{11}$$

III. STOKES' THEOREM

Maxwell's equations can be put into their integral forms through the use of the *Kelvin–Stokes theorem*:

$$\int_{\Omega} \mathrm{d}^2\boldsymbol{a} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) = \oint_{\partial\Omega} \mathrm{d}\boldsymbol{l} \cdot \boldsymbol{F} \tag{12}$$

and the *divergence theorem*:

$$\int_{\Omega} \mathrm{d}^3r \, \boldsymbol{\nabla} \cdot \boldsymbol{F} = \oint_{\partial\Omega} \mathrm{d}^2r \, \boldsymbol{F} \cdot \hat{\boldsymbol{n}}. \tag{13}$$

Here, Ω is some orientable manifold and $\partial\Omega$ is its boundary.

You do not need to know this for the exam, but both of these equations are special cases of the generalized Stokes' theorem, which can be expressed as:

$$\int_{\partial\Omega} \omega = \int_{\Omega} \mathrm{d}\omega \, . \tag{14}$$

Here, ω is a differential form and $\mathrm{d}\omega$ is the exterior derivative of ω.

IV. HELMHOLTZ DECOMPOSITION

Given the divergence and curl of a vector field $\boldsymbol{F}$ over $\mathbb{R}^3$, one can reconstruct $\boldsymbol{F}$ through the formula:

$$\boldsymbol{F}(\boldsymbol{r}) = -\frac{1}{4\pi}\boldsymbol{\nabla} \int \mathrm{d}^3r' \, \frac{\boldsymbol{\nabla} \cdot \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} + \frac{1}{4\pi}\boldsymbol{\nabla} \times \int \mathrm{d}^3r' \, \frac{\boldsymbol{\nabla} \times \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{15}$$

From Maxwell's equations and the Helmholtz decomposition, one finds that in Lorenz gauge:

$$\left(\nabla^2 - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right)\phi = -\frac{\rho}{\epsilon_0}, \tag{16}$$

$$\left(\nabla^2 - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{A} = -\mu_0 \boldsymbol{J}. \tag{17}$$

V. GREEN'S FUNCTIONS

To invert Eq. (16) and Eq. (17), one must find the *Green's function* that satisfies:

$$\left(\nabla^2 - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right) G\left(\boldsymbol{r}, t\right) = \delta^3\left(\boldsymbol{r}\right)\delta\left(t\right). \tag{18}$$

Then, the solution to:

$$\left(\nabla^2 - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right) f = g \tag{19}$$

is given by:

$$f\left(\boldsymbol{r}, t\right) = \int \mathrm{d}t \int \mathrm{d}^3 r\, g\left(\boldsymbol{r}', t'\right) G\left(\boldsymbol{r} - \boldsymbol{r}', t - t'\right), \tag{20}$$

which can be verified by taking $\nabla^2 - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}$ of both sides. It turns out that the solution to Eq. (18) is given by:

$$G\left(\boldsymbol{r}, t\right) = -\frac{1}{4\pi}\frac{\delta\left(t - \frac{r}{c}\right)}{r}. \tag{21}$$

Doing the time integral in Eq. (20) then finally gives the inverted equations:

$$\phi\left(\boldsymbol{r}, t\right) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3 r'\, \frac{\rho\left(\boldsymbol{r}', t - \frac{r'}{c}\right)}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|}, \tag{22}$$

$$\boldsymbol{A}\left(\boldsymbol{r}, t\right) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3 r'\, \frac{\boldsymbol{J}\left(\boldsymbol{r}', t - \frac{r'}{c}\right)}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|}. \tag{23}$$

VI. CONSERVATION LAWS AND THE MAXWELL STRESS TENSOR

For a differential rate of charge creation s, we have the conservation law:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = s. \tag{24}$$

For charges in electromagnetism $s = 0$, and we have the charge conservation law:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0. \tag{25}$$

Similarly, for the electromagnetic energy density,

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{S} = -\boldsymbol{J}\cdot\boldsymbol{E}, \tag{26}$$

where u is the energy density:

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) \tag{27}$$

and $\boldsymbol{S}$ is the *Poynting vector*:

$$\boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}. \tag{28}$$

This conservation law is sometimes called *Poynting's theorem.* $\boldsymbol{J} \cdot \boldsymbol{E}$ is a differential version of the IV power dissipation you would see in Ohm's law, and $\boldsymbol{S}$ represents differential energy flow.

Finally, for the electromagnetic momentum density,

$$\epsilon_0 \mu_0 \frac{\partial \boldsymbol{S}}{\partial t} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = -\boldsymbol{f}, \tag{29}$$

where $\boldsymbol{S}$ is once again the Poynting vector (such that the first term represents the momentum due to the force given by the energy flux given by $\boldsymbol{S}$), $-\boldsymbol{f}$ is the local creation of electromagnetic momentum due to the Lorentz force law as given in Eq. (5), and $-\boldsymbol{\sigma}$ is the electromagnetic momentum flux, given by the *Maxwell stress tensor*:

$$\boldsymbol{\sigma} = \epsilon_0 \left(\boldsymbol{E} \otimes \boldsymbol{E} - \frac{E^2}{2} \boldsymbol{I} \right) + \frac{1}{\mu_0} \left(\boldsymbol{B} \otimes \boldsymbol{B} - \frac{B^2}{2} \boldsymbol{I} \right). \tag{30}$$

In lecture, $\boldsymbol{\sigma}$ is denoted $\boldsymbol{T}$; here, I use $\boldsymbol{\sigma}$ to avoid confusion with the *electromagnetic stress energy tensor.* You will not need to know it for the exam, but it conveniently combines the energy density and momentum density conservation laws into a single tensor. See the notes from the second recitation section if you want to learn more about it.