8.311 Recitation Notes

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I. INTRODUCTION

In lecture, you derived the Liénard–Wiechert potential directly from the equation for the potentials in terms of the currents. Here, we give an alternate derivation that makes manifest the connection between the Liénard–Wiechert potential and special relativity.

II. TRANSFORMING THE FIELDS

Consider a charged particle of charge q in its rest frame, located at $\boldsymbol{r_q}$. Then, in Lorenz gauge, the potentials are:

$$\phi\left(\boldsymbol{r}, t\right) = \frac{q}{4\pi\epsilon_0 \left\|\boldsymbol{r} - \boldsymbol{r_q}\left(t_r\left(t\right)\right)\right\|}, \tag{1}$$

$$\boldsymbol{A}\left(\boldsymbol{r}, t\right) = 0, \tag{2}$$

where t_r is the retarded time:

$$t_r = t - \frac{\left\|\boldsymbol{r} - \boldsymbol{r_q}\left(t_r\right)\right\|}{\mathrm{c}}. \tag{3}$$

How do these fields transform under a boost in the x-direction? Remembering that ϕ and $\boldsymbol{A}$ are really just components of a single four-vector

$$A^\mu = \begin{pmatrix} \frac{\phi}{\mathrm{c}} \\ \boldsymbol{A} \end{pmatrix}^\mu, \tag{4}$$

we can consider the pure Lorentz boost corresponding to changing to a frame with velocity $-v_x \hat{\boldsymbol{e}}_{\boldsymbol{x}}$ (corresponding to the charged particle having a velocity $v_x \hat{\boldsymbol{e}}_{\boldsymbol{x}}$):

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\left(w\right) & \sinh\left(w\right) & 0 & 0 \\ \sinh\left(w\right) & \cosh\left(w\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^\mu{}_\nu, \tag{5}$$

where w is the rapidity:

$$\cosh\left(w\right) = \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{\mathrm{c}^2}}}. \tag{6}$$

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Under this transformation,

$$\begin{aligned}
A^{\mu} \mapsto {} & \Lambda^{\mu}{}_{\nu} A^{\nu} \\
= {} & \begin{pmatrix} \frac{\phi}{\mathrm{c}} \cosh\left(w\right) + A_x \sinh\left(w\right) \\ A_x \cosh\left(w\right) + \frac{\phi}{\mathrm{c}} \sinh\left(w\right) \\ \boldsymbol{A}_{\perp} \end{pmatrix}^{\mu} \\
= {} & \begin{pmatrix} \gamma \left(\frac{\phi}{c} + \beta_x A_x\right) \\ \gamma \left(A_x + \beta_x \frac{\phi}{\mathrm{c}}\right) \\ \boldsymbol{A}_{\perp} \end{pmatrix}^{\mu} \\
= {} & \begin{pmatrix} \gamma \frac{\phi}{c} \\ \gamma \beta_x \frac{\phi}{\mathrm{c}} \\ \boldsymbol{A}_{\perp} \end{pmatrix}^{\mu} ,
\end{aligned} \tag{7}$$

where:

$$\boldsymbol{\beta} \equiv \frac{\boldsymbol{v}}{\mathrm{c}}. \tag{8}$$

So, in conclusion, generalizing to boosts in arbitrary directions gives:

$$\phi'\left(\boldsymbol{r}, t\right) = \frac{1}{\sqrt{1 - \frac{v^2}{\mathrm{c}^2}}} \phi\left(\boldsymbol{r}, t\right), \tag{9}$$

$$\boldsymbol{A}'\left(\boldsymbol{r}, t\right) = \frac{\boldsymbol{\beta}}{\sqrt{\mathrm{c}^2 - v^2}} \phi\left(\boldsymbol{r}, t\right). \tag{10}$$

Now, all that's left is to transform the coordinates.

III. TRANSFORMING THE COORDINATES

How does $\boldsymbol{r}$ transform? Using the inverse Lorentz transform:

$$\left(\Lambda^{-1}\right)^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\left(w\right) & -\sinh\left(w\right) & 0 & 0 \\ -\sinh\left(w\right) & \cosh\left(w\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mu}{}_{\nu} , \tag{11}$$

we have that:

$$\begin{aligned}
(r')^{\mu} &\mapsto \left(\Lambda^{-1}\right)^{\mu}_{\ \nu} (r')^{\nu} \\
&= \begin{pmatrix} \mathrm{c}t' \cosh(w) - x' \sinh(w) \\ x' \cosh(w) - \mathrm{c}t' \sinh(w) \\ \boldsymbol{r'_{\perp}} \end{pmatrix}^{\mu} \\
&= \begin{pmatrix} \gamma\left(\mathrm{c}t' - \frac{v_x}{\mathrm{c}} x'\right) \\ \gamma\left(x' - v_x t'\right) \\ \boldsymbol{A}_{\perp} \end{pmatrix}^{\mu} .
\end{aligned} \tag{12}$$

That is,

$$t = \gamma\left(\mathrm{c}t' - \frac{v_x}{\mathrm{c}} x'\right), \tag{13}$$

$$x = \gamma\left(x' - v_x t'\right). \tag{14}$$

Therefore,

$$\begin{aligned}
\phi(\boldsymbol{r'}, t') &= \frac{q}{4\pi\epsilon_0 \left\|\boldsymbol{r}(\boldsymbol{r'}) - \boldsymbol{r_q}\left(\boldsymbol{r'_q}, t'_r(t')\right)\right\|} \\
&= \frac{q}{4\pi\epsilon_0 \gamma \left|x' - v_x t' - \left(x'_q - v_x t'_r(t')\right)\right|} \\
&= \frac{q}{4\pi\epsilon_0 \gamma \left|x' - x'_q - v_x\left(t' - t'_r(t')\right)\right|}.
\end{aligned} \tag{15}$$

Since $t' - t'_r(t')$ is just the time delay $\frac{\left|x' - x'_q(t'_r)\right|}{\mathrm{c}}$, we finally have that (once again generalizing to arbitrary boosts):

$$\phi(\boldsymbol{r'}, t') = \frac{q}{4\pi\epsilon_0 \gamma \left(1 - \boldsymbol{\beta} \cdot \boldsymbol{\hat{n}}\left(t'_r(t')\right)\right) \left\|\boldsymbol{r'} - \boldsymbol{r'_q}\left(t'_r(t')\right)\right\|}, \tag{16}$$

where:

$$\boldsymbol{\hat{n}}\left(t'_r(t')\right) = \frac{\boldsymbol{r'} - \boldsymbol{r'_q}\left(t'_r(t')\right)}{\left\|\boldsymbol{r'} - \boldsymbol{r'_q}\left(t'_r(t')\right)\right\|}. \tag{17}$$

To simplify notation, this is often written as:

$$\phi(\boldsymbol{r'}, t') = \left(\frac{q}{4\pi\epsilon_0 \gamma \left(1 - \boldsymbol{\beta} \cdot \boldsymbol{\hat{n}}\right) \left\|\boldsymbol{r'} - \boldsymbol{r'_q}\right\|}\right)_{t'_r(t')}. \tag{18}$$

We therefore finally have that:

$$\phi'(\boldsymbol{r'}, t') = \left(\frac{q}{4\pi\epsilon_0 \left(1 - \boldsymbol{\beta} \cdot \boldsymbol{\hat{n}}\right) \left\|\boldsymbol{r'} - \boldsymbol{r'_q}\right\|}\right)_{t'_r(t')}, \tag{19}$$

$$\boldsymbol{A}'(\boldsymbol{r'}, t') = \left(\frac{\mu_0 q \boldsymbol{\beta}}{4\pi \left(1 - \boldsymbol{\beta} \cdot \boldsymbol{\hat{n}}\right) \left\|\boldsymbol{r'} - \boldsymbol{r'_q}\right\|}\right)_{t'_r(t')}. \tag{20}$$