# 8.311 Recitation Notes

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## I. INTRODUCTION

Today I am going to talk about the microscopic origins of Ohm's law—namely, the Drude model.

## II. THE DRUDE MODEL IN A DC FIELD

Let us first show the "standard" Ohm's law; namely, given a static potential difference $V$ and resistance $R$ in a circuit, the current $I$ satisfies the relation:

$$V = IR. \tag{1}$$

To see this, let us consider an electron bouncing against stationary ions in a wire. Assume collions occur a time $\tau$ apart on average. Then, on average, the electron will have gained a momentum:

$$\Delta \langle \boldsymbol{p} \rangle = -\mathrm{e}\tau \boldsymbol{E} \tag{2}$$

between two collisions occuring at times $t - \tau$ and $t$. Making $\Delta \langle \boldsymbol{p} \rangle$ explicit, we have that:

$$\langle \boldsymbol{p}(t) \rangle - \langle \boldsymbol{p}(t-\tau) \rangle = -\mathrm{e}\tau \boldsymbol{E} \tag{3}$$

for all $t$. Assuming forward collisions are just as likely as backwards collisions, the average momentum immediately after a collision $\langle \boldsymbol{p}(t-\tau) \rangle = 0$; therefore,

$$\langle \boldsymbol{p}(t) \rangle = -\mathrm{e}\tau \boldsymbol{E}. \tag{4}$$

As $\langle \boldsymbol{p} \rangle = \mathrm{m_e} \langle \boldsymbol{v} \rangle$ and $\langle \boldsymbol{J} \rangle = -\mathrm{e} n_e \langle \boldsymbol{v} \rangle$, we therefore have that:

$$\boldsymbol{J} = \frac{\mathrm{e}^2 n_e \tau}{\mathrm{m_e}} \boldsymbol{E}. \tag{5}$$

This microscopic model of resistance is called the *Drude model*.

Taking $A$ to be the cross-sectional area of the wire and $L$ the length of the circuit, we therefore have that:

$$\begin{aligned} I &= \frac{\mathrm{e}^2 n_e \tau A}{\mathrm{m_e} L} V \\ \implies V &= I \frac{\mathrm{m_e} L}{\mathrm{e}^2 n_e \tau A}. \end{aligned} \tag{6}$$

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Thus, in the Drude model in a DC field,

$$R = \frac{\mathrm{m_e} L}{\mathrm{e}^2 n_e \tau A}. \tag{7}$$

Sometimes this is written in terms of the resistivity

$$\rho = \frac{\mathrm{m_e}}{\mathrm{e}^2 n_e \tau}, \tag{8}$$

such that:

$$R = \frac{L\rho}{A}. \tag{9}$$

## III. THE DRUDE MODEL IN AN AC FIELD

What does the Drude model predict in an AC field? To analyze this situation, let us be more careful with the time coordinate of our DC analysis. Namely, at a time $t + \mathrm{d}t$, we have that:

$$\langle \boldsymbol{p} \rangle (t + \mathrm{d}t) = \left(1 - \frac{\mathrm{d}t}{\tau}\right) \left(\langle \boldsymbol{p} \rangle (t) - \mathrm{e}\boldsymbol{E}\, \mathrm{d}t\right), \tag{10}$$

where the factor of $1 - \frac{\mathrm{d}t}{\tau}$ is the fraction of particles on average that have not collided in a time $\mathrm{d}t$ (and the ones that have would contribute in a higher order of $\mathrm{d}t$). This results in the differential equation:

$$\frac{\mathrm{d}\langle \boldsymbol{p} \rangle}{\mathrm{d}t} = -\left(\mathrm{e}\boldsymbol{E} + \frac{\langle \boldsymbol{p} \rangle}{\tau}\right). \tag{11}$$

Considering time dependences of the form $\boldsymbol{p}(t) = \boldsymbol{p_0}\mathrm{e}^{-\mathrm{i}\omega t}$, $\boldsymbol{E}(t) = \boldsymbol{E_0}\mathrm{e}^{-\mathrm{i}\omega t}$, and $\boldsymbol{J}(t) = \boldsymbol{J_0}\mathrm{e}^{-\mathrm{i}\omega t}$, we therefore have that:

$$\begin{aligned} \mathrm{i}\omega \langle \boldsymbol{p_0} \rangle &= \mathrm{e}\boldsymbol{E_0} + \frac{\langle \boldsymbol{p_0} \rangle}{\tau} \\ \implies \boldsymbol{p_0} &= \frac{\mathrm{e}}{\mathrm{i}\omega - \frac{1}{\tau}} \boldsymbol{E_0} \\ \implies \boldsymbol{J_0} &= \frac{\mathrm{e}^2 n_e \tau}{\mathrm{m_e}\left(1 - \mathrm{i}\omega\tau\right)} \boldsymbol{E_0}. \end{aligned} \tag{12}$$

This recovers Eq. (5) for $\omega = 0$.

What does the complex part of Eq. (12) tell us? From Ampère's law (and taking $\boldsymbol{B}(t) =$

$\boldsymbol{B_0}\mathrm{e}^{-\mathrm{i}\omega t}$), we have that:

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{B} &= \mu_0 \left( \boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \\
\implies \boldsymbol{\nabla} \times \boldsymbol{B}_0 &= \mu_0 \left( \frac{\mathrm{e}^2 n_e \tau}{\mathrm{m_e}\,(1 - \mathrm{i}\omega\tau)} \boldsymbol{E_0} - \mathrm{i}\epsilon_0 \omega \boldsymbol{E_0} \right) \\
&= -\frac{\mathrm{i}\omega}{\mathrm{c}^2} \left( 1 + \frac{\mathrm{i}\mathrm{e}^2 n_e \tau}{\epsilon_0 \omega \mathrm{m_e}\,(1 - \mathrm{i}\omega\tau)} \right) \boldsymbol{E_0} \\
\implies \boldsymbol{\nabla} \times \boldsymbol{B} &= \mu_0 \epsilon_0 \epsilon_r \frac{\partial \boldsymbol{E}}{\partial t},
\end{aligned} \tag{13}$$

where:

$$\epsilon_r = 1 + \frac{\mathrm{i}\mathrm{e}^2 n_e \tau}{\epsilon_0 \omega \mathrm{m_e}\,(1 - \mathrm{i}\omega\tau)}. \tag{14}$$

$\epsilon_r$ is the *relative permittivity*, and gives the dielectric polarization density $\boldsymbol{P}_{\text{conductivity}}$ of the medium to be:

$$\boldsymbol{P}_{\text{conductivity}} = \frac{\mathrm{i}\mathrm{e}^2 n_e \tau}{\omega \mathrm{m_e}\,(1 - \mathrm{i}\omega\tau)} \boldsymbol{E}. \tag{15}$$

$\boldsymbol{P}_{\text{conductivity}}$ is the density of induced dipole moments in the medium due to the nonzero conductivity of the material. Therefore, the Drude model predicts a particular polarization response of conductive media in the presence of an AC electric field; experiments show that Eq. (15) is indeed (approximately) true! It turns out the Drude model is very good at modeling these effects, and really only begins to break down when predicting temperature dependences due to quantum effects.