8.311 Recitation Notes

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I. INTRODUCTION

Today I am going to review the material that will be on your second exam. The exam will cover the material in problem sets four, five, and six, and not include topics like relativity.

II. DIPOLE RADIATION

We recall the potentials in terms of the currents:

$$\phi(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3 r' \, \frac{\rho(\boldsymbol{r'}, t_{\text{ret}}(\boldsymbol{r'}, t))}{\|\boldsymbol{r} - \boldsymbol{r'}\|}, \tag{1}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3 r' \, \frac{\boldsymbol{J}(\boldsymbol{r'}, t_{\text{ret}}(\boldsymbol{r'}, t))}{\|\boldsymbol{r} - \boldsymbol{r'}\|}, \tag{2}$$

where:

$$t_{\text{ret}}(\boldsymbol{r'}, t) = t - \frac{\|\boldsymbol{r} - \boldsymbol{r'}\|}{\mathrm{c}}. \tag{3}$$

For $\|\boldsymbol{r} - \boldsymbol{r'}\| \ll \mathrm{c}t$, we can Taylor expand the currents. Assuming we have a time-varying electric dipole $\boldsymbol{p}$, this expansion gives:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \left(\frac{q\hat{\boldsymbol{n}}}{r^2} + \frac{3\hat{\boldsymbol{n}}(\boldsymbol{p}(t') \cdot \hat{\boldsymbol{n}} - \boldsymbol{p}(t'))}{r^3} + \frac{3\hat{\boldsymbol{n}}(\dot{\boldsymbol{p}}(t') \cdot \hat{\boldsymbol{n}} - \dot{\boldsymbol{p}}(t'))}{\mathrm{c}r^2} + \frac{(\ddot{\boldsymbol{p}}(t') \times \hat{\boldsymbol{n}}) \times \hat{\boldsymbol{n}}}{\mathrm{c}^2 r} \right)_{t' = t - \frac{r}{\mathrm{c}}}, \tag{4}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \left(\left(\frac{\dot{\boldsymbol{p}}(t')}{r^2} + \frac{\ddot{\boldsymbol{p}}(t')}{\mathrm{c}r} \right) \times \hat{\boldsymbol{n}} \right)_{t' = t - \frac{r}{\mathrm{c}}}. \tag{5}$$

The terms with no time derivatives are called the quasi-static terms, the terms with a single time derivative the induction terms, and the terms with double time derivatives the radiation terms. They become important in different regimes of r. The radiation term is what carries electromagnetic energy to infinite r, giving:

$$P = \frac{\ddot{p}^2}{6\pi\epsilon_0 \mathrm{c}^3} \tag{6}$$

Repeating the same analysis for a time-varying magnetic dipole $\boldsymbol{m}$ yields:

$$\boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\mu_0}{4\pi} \left(\left(\frac{\dot{\boldsymbol{m}}(t')}{r^2} + \frac{\ddot{\boldsymbol{m}}(t')}{\mathrm{c}r} \right) \times \hat{\boldsymbol{n}} \right)_{t' = t - \frac{r}{\mathrm{c}}}, \tag{7}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \left(\frac{3\hat{\boldsymbol{n}}(\boldsymbol{m}(t') \cdot \hat{\boldsymbol{n}} - \boldsymbol{m}(t'))}{r^3} + \frac{3\hat{\boldsymbol{n}}(\dot{\boldsymbol{m}}(t') \cdot \hat{\boldsymbol{n}} - \dot{\boldsymbol{m}}(t'))}{\mathrm{c}r^2} + \frac{(\ddot{\boldsymbol{m}}(t') \times \hat{\boldsymbol{n}}) \times \hat{\boldsymbol{n}}}{\mathrm{c}^2 r} \right)_{t' = t - \frac{r}{\mathrm{c}}}. \tag{8}$$

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Once again, the radiation term is what carries electromagnetic energy to infinite r, giving:

$$P = \frac{\mu_0 \ddot{m}^2}{6\pi c^3} \tag{9}$$

III. THE LIÉNARD–WIECHERT POTENTIALS

The analysis of Sec. II was done for nonrelativistic dipoles. How does the analysis change for relativistic point charges of charge q? Doing a bunch of ugly math gives the *Liénard–Wiechert potentials*:

$$\phi(\boldsymbol{r}, t) = \frac{q}{4\pi\epsilon_0 \left\|\boldsymbol{r} - \boldsymbol{r_q}(t_{\text{ret}})\right\| \left(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta}(t_{\text{ret}})\right)}, \tag{10}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0 q \boldsymbol{v}_q(t_{\text{ret}})}{4\pi \left\|\boldsymbol{r} - \boldsymbol{r_q}(t_{\text{ret}})\right\| \left(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta}(t_{\text{ret}})\right)}, \tag{11}$$

for a charge at position $\boldsymbol{r}_q$, where:

$$\boldsymbol{\beta} = \frac{\boldsymbol{v_q}}{c}. \tag{12}$$

The factors of $\frac{1}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}}$ you can think of as coming from special relativity, and they disappear in the nonrelativistic limit.

Using these expressions for the potentials, one can find the expressions for the fields:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{q}{4\pi\epsilon_0} \left(\frac{\left(1 - \beta(t_{\text{ret}})^2\right)\left(\hat{\boldsymbol{n}} - \boldsymbol{\beta}(t_{\text{ret}})\right)}{\left\|\boldsymbol{r} - \boldsymbol{r}_q(t_{\text{ret}})\right\|^2 \left(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta}(t_{\text{ret}})\right)^3} + \frac{\hat{\boldsymbol{n}} \times \left(\left(\hat{\boldsymbol{n}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right)}{c \left\|\boldsymbol{r} - \boldsymbol{r}_q(t_{\text{ret}})\right\| \left(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta}(t_{\text{ret}})\right)^3} \right), \tag{13}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c}\left(\hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r}, t')\right)_{t'=t_{\text{ret}}}. \tag{14}$$

Calculating the power radiated away by these fields yields:

$$\frac{\mathrm{d}W}{\mathrm{d}\Omega\, \mathrm{d}t_{\text{ret}}} = \frac{q^2}{(4\pi)^2 \epsilon_0 c \left(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta}\right)^5} \left\| \hat{\boldsymbol{n}} \times \left(\left(\hat{\boldsymbol{n}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right) \right\|^2. \tag{15}$$

IV. THE ABRAHAM–LORENTZ FORCE

When moving a charged object, the charged object interacts with the fields it generates to create a force on itself; this is called the *Abraham–Lorentz force*. By calculating the fields due to the charged object and then evaluating the resulting Lorentz force on the charged object, one finds that:

$$\boldsymbol{F}_{\text{AL}} = -\frac{4U_E}{3c^2}\boldsymbol{a} + \frac{q^2}{6\pi\epsilon_0 c^3}\dot{\boldsymbol{a}}, \tag{16}$$

where:

$$U_E = \frac{1}{2} \int \mathrm{d}^3x \, \phi\left(\boldsymbol{r}\right) \rho\left(\boldsymbol{r}\right) \tag{17}$$

is the energy required to assemble the charge configuration. The second of these terms can be thought of as a radiation reaction force due to the power radiated away by the moving charge to infinity. The first term is an electrmagnetic inertia term, which is a local phenomenon.