8.311 Recitation Notes

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I. INTRODUCTION

Today I am going to emphasize the connection between relativity and electromagnetism. In the second recitation I talked about how they were mathematically linked, but today I will (hopefully) give more physical intuition for it.

II. FIELD TRANSFORMATIONS

Let us consider a charged particle of velocity $\boldsymbol{v} = v_x \hat{\boldsymbol{e}}_{\boldsymbol{x}}$, and a pure Lorentz boost into the frame the charged particle is stationary:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh(w) & \sinh(w) & 0 & 0 \\ \sinh(w) & \cosh(w) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mu}_{\ \ \nu} , \tag{1}$$

where w is the rapidity:

$$\cosh(w) = \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2}$$

and due to us boosting by $-\boldsymbol{v}$:

$$\sinh(w) = -\gamma \frac{\boldsymbol{v}}{c} = -\frac{\boldsymbol{v}}{c\sqrt{1 - \frac{v^2}{c^2}}}. \tag{3}$$

How do the fields change under such a boost? Recalling that the fields transform as components of the electromagnetic tensor:

$$F^{\mu}{}_{\nu} = \begin{pmatrix} 0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\ \frac{E_1}{c} & 0 & B_3 & -B_2 \\ \frac{E_2}{c} & -B_3 & 0 & B_1 \\ \frac{E_3}{c} & B_2 & -B_1 & 0 \end{pmatrix}^{\mu}_{\ \ \nu} , \tag{4}$$

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we know that:

$$
\begin{aligned}
(F')^{\mu}{}_{\nu} &= \Lambda^{\mu}{}_{\alpha} F^{\alpha}{}_{\beta} \left(\Lambda^{-1}\right)^{\beta}{}_{\nu} \\
\iff \begin{pmatrix} 0 & \frac{E'_1}{\mathrm{c}} & \frac{E'_2}{\mathrm{c}} & \frac{E'_3}{\mathrm{c}} \\ \frac{E'_1}{\mathrm{c}} & 0 & B'_3 & -B'_2 \\ \frac{E'_2}{\mathrm{c}} & -B'_3 & 0 & B'_1 \\ \frac{E'_3}{\mathrm{c}} & B'_2 & -B'_1 & 0 \end{pmatrix}^{\mu}{}_{\nu} &= \begin{pmatrix} \cosh(w) & \sinh(w) & 0 & 0 \\ \sinh(w) & \cosh(w) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mu}{}_{\alpha} \begin{pmatrix} 0 & \frac{E_1}{\mathrm{c}} & \frac{E_2}{\mathrm{c}} & \frac{E_3}{\mathrm{c}} \\ \frac{E_1}{\mathrm{c}} & 0 & B_3 & -B_2 \\ \frac{E_2}{\mathrm{c}} & -B_3 & 0 & B_1 \\ \frac{E_3}{\mathrm{c}} & B_2 & -B_1 & 0 \end{pmatrix}^{\alpha}{}_{\beta} \\
&\times \begin{pmatrix} \cosh(w) & -\sinh(w) & 0 & 0 \\ -\sinh(w) & \cosh(w) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\beta}{}_{\nu} .
\end{aligned} \tag{5}
$$

Therefore, the field components in the frame where the electron is stationary are:

$$E'_x = E_x, \tag{6}$$

$$E'_y = \gamma\left(E_y - v_x B_z\right), \tag{7}$$

$$E'_z = \gamma\left(E_z + v_x B_y\right), \tag{8}$$

$$B'_x = B_x, \tag{9}$$

$$B'_y = \gamma\left(B_y + \frac{v_x}{\mathrm{c}^2} E_z\right), \tag{10}$$

$$B'_z = \gamma\left(B_z - \frac{v_x}{\mathrm{c}^2} E_y\right). \tag{11}$$

That is, for $v \ll \mathrm{c}$,

$$\boldsymbol{E'} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}, \tag{12}$$

$$\boldsymbol{B'} = \boldsymbol{B} - \frac{\boldsymbol{v} \times \boldsymbol{E}}{\mathrm{c}^2}. \tag{13}$$

III. CONSISTENCY OF THE LORENTZ FORCE LAW

From here, it is easy to see why the Lorentz force law has a $\boldsymbol{v} \times \boldsymbol{B}$ term. The Lorentz force in terms of what electric field the electron "feels" is:

$$\boldsymbol{F} = q\boldsymbol{E'}. \tag{14}$$

Transforming to the lab frame then gives from Eq. (12):

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right). \tag{15}$$