8.311 Recitation Notes

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I. INTRODUCTION

Today I will talk about the general solution to waves traveling in an anisotropic medium.

II. THE DISPERSION RELATION

Fourier analyzing Ampère's circuital law with Maxwell's correction gives:

$$\mathrm{i}\boldsymbol{k} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} - \frac{\mathrm{i}\omega}{\mathrm{c}^2} \boldsymbol{E}. \tag{1}$$

Similarly, Fourier analyzing Faraday's law gives:

$$\mathrm{i}\boldsymbol{k} \times \boldsymbol{E} = \mathrm{i}\omega \boldsymbol{B}. \tag{2}$$

Combining these results yields:

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}) + \frac{\omega^2}{\mathrm{c}^2} \boldsymbol{E} + \mathrm{i}\mu_0 \omega \boldsymbol{J} = \boldsymbol{0}. \tag{3}$$

Using Ohm's law

$$\boldsymbol{J} = \boldsymbol{\sigma} \boldsymbol{E} \tag{4}$$

then gives:

$$\left(\boldsymbol{k} \otimes \boldsymbol{k} - k^2 + \frac{\omega^2}{\mathrm{c}^2} + \mathrm{i}\mu_0 \omega \boldsymbol{\sigma} \right) \boldsymbol{E} = \boldsymbol{0}. \tag{5}$$

What is $\boldsymbol{\sigma}$ in terms of the usual $\boldsymbol{\epsilon}$? Assuming the only current is bound, we have from Fourier analyzing Ampère's circuital law with Maxwell's correction that:

$$\frac{\mathrm{i}}{\mu_0} \boldsymbol{k} \times \boldsymbol{B} = -\mathrm{i}\omega \boldsymbol{\epsilon} \boldsymbol{E}; \tag{6}$$

that is, it is equivalent when taking:

$$\boldsymbol{\epsilon} = \epsilon_0 + \frac{\mathrm{i}\boldsymbol{\sigma}}{\omega}. \tag{7}$$

We can therefore rewrite Eq. (5) as:

$$\left(\boldsymbol{k} \otimes \boldsymbol{k} - k^2 + \frac{\omega^2}{\epsilon_0 \mathrm{c}^2} \boldsymbol{\epsilon} \right) \boldsymbol{E} = \boldsymbol{0}. \tag{8}$$

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Nontrivial solutions therefore correspond to:

$$\det\left(\boldsymbol{k}\otimes\boldsymbol{k} - k^2 + \frac{\omega^2}{\epsilon_0 \mathrm{c}^2}\boldsymbol{\epsilon}\right) = 0. \tag{9}$$

This equation is the generalized dispersion relation between $\boldsymbol{k}$ and ω. When $\boldsymbol{\epsilon} = \epsilon_0$, this reduces to (assuming a nontrivial time dependence)

$$\frac{\omega}{k} = \pm\mathrm{c}. \tag{10}$$

For an isotropic medium where $\boldsymbol{\epsilon} = \epsilon$, we just have that the index of refraction is:

$$N = \left|\frac{\mathrm{c}k}{\omega}\right| = \sqrt{\frac{\epsilon}{\epsilon_0}}. \tag{11}$$

Furthermore, we can analyze which polarizations have the given dispersion relation. In matrix form in the isotropic case (and taking $\boldsymbol{k} = k\hat{\boldsymbol{e}}_{\boldsymbol{z}}$,

$$\begin{pmatrix} \frac{\omega^2}{\epsilon_0 \mathrm{c}^2}\epsilon - k^2 & 0 & 0 \\ 0 & \frac{\omega^2}{\epsilon_0 \mathrm{c}^2}\epsilon - k^2 & 0 \\ 0 & 0 & \frac{\omega^2}{\epsilon_0 \mathrm{c}^2}\epsilon \end{pmatrix}\boldsymbol{E} = \boldsymbol{0}. \tag{12}$$

For polarizations with an index of refraction given by Eq. (11),

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\omega^2}{\epsilon_0 \mathrm{c}^2}\epsilon \end{pmatrix}\boldsymbol{E} = \boldsymbol{0}. \tag{13}$$

Therefore, we must have that:

$$E^z = 0. \tag{14}$$

Thus, we have derived that no longitudinal polarizations are supported in isotropic mediums with nonzero ϵ.